

A GEOMETRIC CONSTRUCTION OF GENERALIZED q -SCHUR ALGEBRAS

STEPHEN DOTY AND YIQIANG LI

ABSTRACT. We show that the algebras $\mathcal{C}(X \times X)$ in [Li10] and \mathcal{L}_d in [Li12] are generalized q -Schur algebras as defined in [D03]. This provides a geometric construction of generalized q -Schur algebras in types **A**, **D** and **E**. We give a parameterization of Nakajima's Lagrangian quiver variety of type **D** associated to a certain highest weight.

INTRODUCTION

Beilinson, Lusztig, and MacPherson [BLM] gave a geometric construction of the q -Schur algebra in type **A** in terms of the relative position of pairs of flags on a finite dimensional vector space over a finite field of q elements. (See also [Du95].) The question naturally arises: can a similar construction be made in other types? We will show that this question admits a positive answer in types **A**, **D**, and **E**.

Generalized Schur algebras for arbitrary type were introduced by S. Donkin in [Don86], and their q -analogues (the generalized q -Schur algebras) were studied in [L93, 29.2], [DS94], and [D03].

In [Li10], a finite dimensional quotient \mathcal{C} of the quantum algebra of type \mathbf{D}_{m+2} is constructed geometrically by using functions on pairs of ramified partial flag varieties. This algebra enjoys many properties similar to that of generalized q -Schur algebras constructed algebraically in [D03]. We show that the algebra \mathcal{C} is a certain generalized q -Schur algebra of type \mathbf{D}_{m+2} ; hence this generalized q -Schur algebra admits a geometric construction. We first use an argument similar to that in [L03, 2.26, 2.27] to show that there is a surjective algebra homomorphism from a generalized q -Schur algebra in [D03] to \mathcal{C} . Then we apply the fact that (the rational form of) generalized q -Schur algebras are semisimple to obtain the injectivity result.

We also explain how a similar argument can be used to show that the algebra \mathcal{L}_d in [Li12] is isomorphic to the integral form of a generalized q -Schur algebra of finite type. In the final part of this note, we obtain a parametrization of Nakajima's Lagrangian quiver variety of type \mathbf{D}_{m+2} by using the connected components of the ramified partial flag variety studied in this note.

1. THE ALGEBRAS \mathcal{C} , \mathcal{C}°

1.1. Recall from [Li10] that we have the following data.

- A Dynkin graph of type \mathbf{D}_{m+2} :

$$\begin{array}{c}
i \\
| \\
k \text{ --- } j_1 \text{ --- } j_2 \text{ --- } \cdots \text{ --- } j_m,
\end{array}$$

whose vertex set is denoted by I and the associated Cartan matrix is $C = (c_{ab})_{a,b \in I}$.

- A finite field \mathbb{F}_q of q elements.
- A fixed d -dimensional vector space D over \mathbb{F}_q .
- A set X of all ‘ramified’ flags in D of the form

$$0 \subseteq V_{j_m} \subseteq \cdots \subseteq V_{j_1} \subseteq U_i, U_k \subseteq U_{j_1} \subseteq \cdots \subseteq U_{j_m} \subseteq D.$$

- A partition $X = \sqcup_{\nu \in \mathbb{N}[I]} X_\nu$ of X , where X_ν contains all flags $U \in X$ subject to the conditions:

$$\dim U_i = \nu_i, \dim U_k = \nu_k, \dim U_{j_\beta} + \dim V_{j_\beta} = \nu_{j_\beta}, \quad \forall 1 \leq \beta \leq m.$$

Notice that X_ν is empty for all but finitely many ν in $\mathbb{N}[I]$. As in [Li10], we consider the \mathbb{C} -vector space $\mathcal{C}'(X \times X)$ of all \mathbb{C} -valued functions on $X \times X$. The vector space $\mathcal{C}'(X \times X)$ admits an associative algebra structure with the multiplication given by the convolution product:

$$f_1 \circ f_2(U, \tilde{U}) = \sum_{U' \in X} f_1(U, U') f_2(U', \tilde{U}), \quad \forall f_1, f_2 \in \mathcal{C}'(X \times X); U, \tilde{U} \in X.$$

The algebra $\mathcal{C}'(X \times X)$ has a unit $\mathbf{1}$ defined by

$$\mathbf{1}(U, \tilde{U}) = \begin{cases} 1, & \text{if } U = \tilde{U}, \\ 0, & \text{otherwise.} \end{cases}$$

For convenience, we shall write $|V|$ for the dimension of a given vector space V . We shall write “ $U \overset{a}{\subset} \tilde{U}$ ” to denote that U is contained in \tilde{U} and U is one-dimension short of \tilde{U} at the position $a \in I$. The notation “ $U \overset{j_1}{\subset} \tilde{U}, V_{j_1} \overset{1}{\subset} \tilde{V}_{j_1}$ ” denotes a pair (U, \tilde{U}) in $X \times X$ satisfying that $U_a = \tilde{U}_a$ for any $a \in I$, $V_a = \tilde{V}_a$ for any $a \neq j_1$, $V_{j_1} \subset \tilde{V}_{j_1}$ and $\dim V_{j_1} + 1 = \dim \tilde{V}_{j_1}$.

We are mainly interested in the subalgebra $\mathcal{C} \equiv \mathcal{C}'(X \times X)$ generated by the following functions E_a , F_a , and $K_a^{\pm 1}$ for any $a \in I$. For $a = i, k$, the functions E_a and F_a are defined by

$$E_a(U, \tilde{U}) = \begin{cases} \sqrt{q}^{-(|\tilde{U}_{j_1}| - |\tilde{U}_a|)}, & \text{if } U \overset{a}{\subset} \tilde{U}, \\ 0 & \text{otherwise,} \end{cases} \quad F_a(U, \tilde{U}) = \begin{cases} \sqrt{q}^{-(|\tilde{U}_a| - |\tilde{V}_{j_1}|)}, & \text{if } U \overset{a}{\supset} \tilde{U}, \\ 0 & \text{otherwise.} \end{cases}$$

For $a = j_\beta$ and $1 \leq \beta \leq m$, the functions E_a and F_a are defined as follows.

$$\begin{aligned}
(1) \quad E_{j_1}(U, \tilde{U}) &= \begin{cases} \sqrt{q}^{-(|\tilde{U}_i|+|\tilde{U}_k|+|\tilde{V}_{j_1}|+|\tilde{V}_{j_2}|)}, & \text{if } U \overset{j_1}{\subset} \tilde{U}, V_{j_1} \overset{1}{\subset} \tilde{V}_{j_1}, \\ \sqrt{q}^{-(|\tilde{U}_{j_2}|+2|\tilde{V}_{j_1}|+|\tilde{U}_{j_1}|)}, & \text{if } U \overset{j_1}{\subset} \tilde{U}, U_{j_1} \overset{1}{\subset} \tilde{U}_{j_1}, \\ 0, & \text{otherwise.} \end{cases} \\
E_{j_\beta}(U, \tilde{U}) &= \begin{cases} \sqrt{q}^{-(|\tilde{U}_{j_{\beta-1}}|+|\tilde{V}_{j_\beta}|+|\tilde{V}_{j_{\beta+1}}|)}, & \text{if } U \overset{j_\beta}{\subset} \tilde{U}, V_{j_\beta} \overset{1}{\subset} \tilde{V}_{j_\beta}, \\ \sqrt{q}^{-(|\tilde{U}_{j_{\beta+1}}|+2|\tilde{V}_{j_\beta}|+|\tilde{U}_{j_\beta}|+|\tilde{V}_{j_{\beta-1}}|)}, & \text{if } U \overset{j_\beta}{\subset} \tilde{U}, U_{j_\beta} \overset{1}{\subset} \tilde{U}_{j_\beta}, \\ 0, & \text{otherwise.} \end{cases} \\
(2) \quad F_{j_1}(U, \tilde{U}) &= \begin{cases} \sqrt{q}^{-(|\tilde{U}_{j_2}|+2|\tilde{U}_{j_1}|+|\tilde{V}_{j_1}|)}, & \text{if } U \overset{j_1}{\supset} \tilde{U}, V_{j_1} \overset{1}{\supset} \tilde{V}_{j_1}, \\ \sqrt{q}^{-(|\tilde{U}_{j_1}|+|\tilde{U}_i|+|\tilde{U}_k|+|\tilde{V}_{j_2}|)}, & \text{if } U \overset{j_1}{\supset} \tilde{U}, U_{j_1} \overset{1}{\supset} \tilde{U}_{j_1}, \\ 0, & \text{otherwise.} \end{cases} \\
F_{j_\beta}(U, \tilde{U}) &= \begin{cases} \sqrt{q}^{-(|\tilde{U}_{j_{\beta+1}}|+2|\tilde{U}_{j_\beta}|+|\tilde{V}_{j_\beta}|+|\tilde{V}_{j_{\beta-1}}|)}, & \text{if } U \overset{j_\beta}{\supset} \tilde{U}, V_{j_\beta} \overset{1}{\supset} \tilde{V}_{j_\beta}, \\ \sqrt{q}^{-(|\tilde{V}_{j_{\beta+1}}|+|\tilde{U}_{j_\beta}|+|\tilde{U}_{j_{\beta-1}}|)}, & \text{if } U \overset{j_\beta}{\supset} \tilde{U}, U_{j_\beta} \overset{1}{\supset} \tilde{U}_{j_\beta}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

The functions $K_a^{\pm 1}$ are given by

$$(3) \quad K_a^{\pm 1}(U, \tilde{U}) = \begin{cases} \sqrt{q}^{\pm(|\tilde{U}_{j_1}|+|\tilde{V}_{j_1}|+2|\tilde{U}_a|)}, & \text{if } U = \tilde{U}, a = i \text{ or } k, \\ \sqrt{q}^{\pm(|\tilde{U}_{j_2}|+|\tilde{V}_{j_2}|+|\tilde{U}_i|+|\tilde{U}_k|+2|\tilde{U}_{j_1}|+2|\tilde{V}_{j_1}|)}, & \text{if } U = \tilde{U}, a = j_1, \\ \sqrt{q}^{\pm(|\tilde{U}_{j_{\beta+1}}|+|\tilde{V}_{j_{\beta+1}}|+|\tilde{U}_{j_{\beta-1}}|+|\tilde{V}_{j_{\beta-1}}|+2|\tilde{U}_{j_\beta}|+2|\tilde{V}_{j_\beta}|)}, & \text{if } U = \tilde{U}, a = j_\beta, \\ 0 & \text{if } U \neq \tilde{U}, \end{cases}$$

where β runs from 2 to m .

In addition to the above functions, we define

$$\mathbf{1}_\nu(U, \tilde{U}) = \begin{cases} 1, & \text{if } U = \tilde{U} \in X_\nu, X_\nu \text{ nonempty,} \\ 0, & \text{otherwise,} \end{cases}$$

for any $\nu \in \mathbb{N}[I]$. It is clear that $\mathbf{1}_\nu \mathbf{1}_{\nu'} = \delta_{\nu, \nu'} \mathbf{1}_\nu$ and

$$(4) \quad \mathbf{1} = \sum_{\nu \in \mathbb{N}[I]} \mathbf{1}_\nu.$$

Let $\dot{\mathcal{C}}$ be the subalgebra of $\mathcal{C}'(X \times X)$ generated by the functions $\mathbf{1}_\nu$, $E_a \mathbf{1}_\nu$ and $F_a \mathbf{1}_\nu$ for any $\nu \in \mathbb{N}[I]$ and $a \in I$. By (4), we see that the algebra $\dot{\mathcal{C}}$ is unital. This fact implies that we have $\mathcal{C} \subseteq \dot{\mathcal{C}}$. Moreover, we have

Lemma 1.2. $\mathcal{C} = \dot{\mathcal{C}}$.

Proof. We only need to show that $\mathbf{1}_\nu$ is in \mathcal{C} for any $\nu \in \mathbb{N}[I]$. This can be shown by an argument similar to the proof of Lemma 2.24 in [L03]. For the sake of completeness, we shall provide the proof here. Note that one can also prove this Lemma by an argument similar to the proof of Lemma 3.2 (i) in [D03].

Since the functions $\mathbf{1}_\nu$ for any $\nu \in \mathbb{N}[I]$ such that $\mathbf{1}_\nu \neq 0$ are orthogonal idempotents, we have

$$(5) \quad \prod_{a \in I} K_a^{n_a} = \sum_{\nu \in \mathbb{N}[I]} \sqrt{q}^{\sum_{a \in I} n_a b_{a,\nu}} \mathbf{1}_\nu,$$

for any $(n_a) \in \mathbb{Z}^I$ where $b_{a,\nu}$ are the exponents of \sqrt{q} in the definition of K_a . The sum $\sum_{a \in I} n_a b_{a,\nu}$ can be rewritten as

$$\sum_{a \in I} d_a n_a + (n_a) \cdot C\nu,$$

where $C\nu$ is a vector in \mathbb{Z}^I whose a -th component is equal to $\sum_{b \in I} c_{ab} \nu_b$ and the dot is the standard inner product of two vectors. So the identity (5) can be rewritten as

$$(6) \quad \sum_{\nu \in \mathbb{N}[I]} \sqrt{q}^{(n_a) \cdot C\nu} \mathbf{1}_\nu = \sqrt{q}^{-\sum_{a \in I} d_a n_a} \prod_{a \in I} K_a^{n_a}.$$

It is enough to show that we can find a vector $(n_a) \in \mathbb{Z}[I]$ such that $(n_a) \cdot C\nu \neq (n_a) \cdot C\nu'$ for any $\nu \neq \nu'$ such that $\mathbf{1}_\nu$ and $\mathbf{1}_{\nu'}$ are not zero. This is because if such a vector (n_a) exists, we can form together with (4) a linear system from (6) by considering the vectors (cn_a) for $c \in \mathbb{N}$. It is clear that the associated coefficient matrix of the linear system is the Vandermonde matrix. Now by choosing the right number of the integers c , we can get a square Vandermonde matrix which is invertible by our choice of the vector (n_a) . This implies that $\mathbf{1}_\nu$ can be expressed as a linear combination of the functions $\sqrt{q}^{-\sum_{a \in I} d_a n_a} \prod_{a \in I} K_a^{n_a}$.

We return to the proof of the existence of such a vector (n_a) . Since A is positive definite, the vector $C(\nu - \nu')$ is non zero for any $\nu \neq \nu'$. Since there are only finitely many ν such that $\mathbf{1}_\nu$ is nonzero, we see that the collection of vectors $C(\nu - \nu')$ such that $\mathbf{1}_\nu$ and $\mathbf{1}_{\nu'}$ are non zero is finite. A standard argument in linear algebra shows that we can find a vector $(n_a) \in \mathbb{Z}^I$ satisfying the requirement. The Lemma follows. \square

1.3. Let $\mathbf{U}_{\sqrt{q}}$ be the specialization of the quantum algebra of type \mathbf{D}_{m+2} at \sqrt{q} . This is an associative algebra over \mathbb{C} generated by the symbols E_a , F_a and $K_a^{\pm 1}$ for $a \in I$ and subject to the following defining relations.

$$\begin{aligned} K_a K_a^{-1} &= 1, K_a K_b = K_b K_a. \\ K_a E_b &= \sqrt{q}^{c_{ab}} E_b K_a, K_a F_b = \sqrt{q}^{-c_{ab}} F_b K_a. \\ E_a F_b - F_b E_a &= \delta_{ab} \frac{K_a - K_a^{-1}}{\sqrt{q} - \sqrt{q}^{-1}}. \\ E_a^2 E_b - (\sqrt{q} + \sqrt{q}^{-1}) E_a E_b E_a + E_b E_a^2 &= 0; \\ F_a^2 F_b - (\sqrt{q} + \sqrt{q}^{-1}) F_a F_b F_a + F_b F_a^2 &= 0, & \text{if } c_{ab} = -1. \\ E_a E_b = E_b E_a, F_a F_b = F_b F_a & & \text{if } c_{ab} = 0. \end{aligned}$$

Let $L(\lambda)$ be the simple $\mathbf{U}_{\sqrt{q}}$ -module of highest-weight $\lambda = \sum_{a \in I} \lambda_a a \in \mathbb{N}[I]$. This is a $\mathbf{U}_{\sqrt{q}}$ -module generated by a vector ξ_λ and subject to the condition:

$$K_a \xi_\lambda = \sqrt{q}^{\lambda_a} \xi_\lambda, \quad E_a \xi_\lambda = 0, \quad \forall a \in I.$$

We denote by \mathcal{I}_D the two-sided ideal of $\mathbf{U}_{\sqrt{q}}$ consisting of all elements u in $\mathbf{U}_{\sqrt{q}}$ such that $u.L(\lambda) = 0$ for any $\lambda \in \mathbb{N}[I]$ satisfying $\lambda = dj_m - C\nu$ for some $\nu \in \mathbb{N}[I]$. Note that the quotient algebra $\mathbf{U}_{\sqrt{q}}/\mathcal{I}_D$ is a generalized q -Schur algebra studied in [D03] with the saturated set π generated by the dominant weight dj_m (or rather $d\omega_{j_m}$ in the notation of [D03]). Indeed, it can be shown that

$$(7) \quad \pi = \{\lambda \in \mathbb{N}[I] \mid \lambda = dj_m - C\nu, \quad \forall \nu \in \mathbb{N}[I]\}.$$

In the language of [D03], $\pi = \{\lambda \in X^+ \mid \lambda = d\omega_{j_m} - \sum_{a \in I} \nu_a \alpha_a, \nu_a \in \mathbb{N}\}$. Note that π is a finite set. Indeed, a necessary condition for $\nu \in \mathbb{N}[I]$ subject to $dj_m - C\nu \in \mathbb{N}[I]$ is that

$$(8) \quad \nu_i + \nu_k \leq \nu_{j_1} \leq \cdots \leq \nu_{j_m} \leq d.$$

Recall from [Li10] that we have a surjective algebra homomorphism

$$\Phi : \mathbf{U}_{\sqrt{q}} \rightarrow \mathcal{C},$$

sending the generators in $\mathbf{U}_{\sqrt{q}}$ to the respective elements in \mathcal{C} . We have

Lemma 1.4. *The morphism Φ factors through a surjective algebra homomorphism*

$$\Psi : \mathbf{U}_{\sqrt{q}}/\mathcal{I}_D \rightarrow \mathcal{C}.$$

Proof. With respect to the partition $X = \sqcup X_\nu$, the algebra \mathcal{C} admits a decomposition

$$\mathcal{C} = \oplus_{\nu, \bar{\nu}} \mathcal{C}(X_\nu \times X_{\bar{\nu}}),$$

where $\mathcal{C}(X_\nu \times X_{\bar{\nu}}) = \mathcal{C} \cap \mathcal{C}'(X_\nu \times X_{\bar{\nu}})$. It is clear from the definitions that

$$K_a f = \sqrt{q}^{\delta_{a,j_m} d - C\nu} f, \quad \forall a \in I, f \in \mathcal{C}(X_\nu \times X_{\bar{\nu}}).$$

This implies that $\mathcal{I}_D \subseteq \ker(\Phi)$. The Lemma follows. \square

Theorem 1.5. *The algebra homomorphism $\Psi : \mathbf{U}_{\sqrt{q}}/\mathcal{I}_D \rightarrow \mathcal{C}$ is an isomorphism.*

Proof. By Corollary 3.13 in [D03], the algebra $\mathbf{U}_{\sqrt{q}}/\mathcal{I}_D$ has a presentation by generators and relations. The generators are E_a, F_a for any $a \in I$ and 1_μ for $\mu \in W\pi$ where W is the Weyl group of type \mathbf{D}_{m+2} . If $\mu = \sum_{a \in I} \mu_a \alpha_a$ (or $\sum_{a \in I} \mu_a \omega_a$ in [D03]), we see that

$$\Psi(1_\mu) = \mathbf{1}_\nu, \quad \text{where } \mu = dj_m - C\nu.$$

This is guaranteed by comparing the defining relations of $\mathbf{U}_{\sqrt{q}}/\mathcal{I}_D$ in [D03, 1.3] with the definition of $\mathbf{1}_\nu$. By Propositions 3.8 and 3.10 in [D03], we see that the algebra $\mathbf{U}_{\sqrt{q}}/\mathcal{I}_D$ is a finite dimensional semisimple algebra and

$$(9) \quad \mathbf{U}_{\sqrt{q}}/\mathcal{I}_D \simeq \bigoplus_{\lambda \in \pi} \text{End}(L(\lambda)),$$

where π is defined in (7). Since $\mathbf{U}_{\sqrt{q}}/\mathcal{I}_D$ is semisimple, so is \mathcal{C} . This implies that \mathcal{C} has a decomposition similar to (9) where the sum runs over a subset of π . Moreover, the homomorphism Ψ is compatible with such decompositions. In order to show that Ψ is an isomorphism, we only need to show that $\Psi(1_\lambda) = \mathbf{1}_\nu$ is non zero for any $\lambda = dj_m - C\nu \in \pi$. It is reduced to show that the variety X_ν is non empty for any $\lambda = dj_m - C\nu \in \pi$. Note that a necessary condition for $\nu \in \mathbb{N}[I]$ to be in π is (8). For any $\nu \in \mathbb{N}[I]$ subject to the condition (8), the associated variety X_ν is always nonempty from the definition. The proof is complete. \square

2. THE ALGEBRA \mathcal{L}_d

2.1. We shall show that a similar argument proves that when the quiver is symmetric of finite type, i.e., a simply-laced Dynkin diagram, the algebra \mathcal{L}_d in [Li12, 6.5] is also a generalized q -Schur algebra. In this situation, the letter d stands for an element in $\mathbb{N}[I]$ where I is the vertex set of the fixed quiver. Let C denote the Cartan matrix of the underlying graph of the quiver. If we set

$$(10) \quad \pi = \{\mu \in \mathbb{N}[I] \mid \mu = d - C\nu, \quad \nu \in \mathbb{N}[I]\}.$$

Then a similar proof as that of Lemma 1.4 shows that the algebra homomorphism Ψ_d in [Li12, 6.6] factors through the integral form ${}_{\mathcal{A}}\mathbf{S}(\pi)$ in [D03, 8.1, 8.2] of the generalized q -Schur algebra $\mathbf{S}(\pi)$ determined by the Cartan matrix C and the saturated set π . Moreover the induced algebra homomorphism ${}_{\mathcal{A}}\mathbf{S}(\pi) \rightarrow \mathcal{L}_d$ is surjective and sends the generators 1_μ to the isomorphism class of the complex of sheaves $\mathcal{S}_{\mu'}$ in [Li12, 5.1] where $\mu' = d - \nu$ if $\mu = d - C\nu$.

Just like the proof of Theorem 1.5, we only need to show that $\mathcal{S}_{\mu'}$ is non zero for any $\mu \in \pi$. Now the condition $\mu = d - C\nu \in \pi$ is equivalent to the condition

$$d_i + \sum_j \nu_j - 2\nu_i \geq 0, \quad \forall i \in I.$$

where the sum runs over all $j \neq i$ such that $c_{ji} = -1$. This immediately implies that $d_i + \sum_j \nu_j \geq \nu_i$ for any $i \in I$. The latter condition guarantees that there is a nonzero element P in $\mathcal{D}_{\mathbf{G}}(\mathbf{E}_\Omega(D, V))$ in [Li12, 6.8] such that the I -graded dimension of V is ν . By the definition of $\mathcal{S}_{\mu'}$, we have $\mathcal{S}_{\mu'}P = P \neq 0$. This property rules out the possibility of $\mathcal{S}_{\mu'} = 0$. In summary, we have proved

Theorem 2.2. *The algebra \mathcal{L}_d in [Li12, 6.5] is the integral form ${}_{\mathcal{A}}\mathbf{S}(\pi)$ of a generalized q -Schur algebra in [D03, 8.1, 8.2] (see also [DS94]) where π is defined in (10).*

Remark. If we choose the quiver such that the associated Cartan matrix C is of type \mathbf{D}_{m+2} , i.e., the same as that of Section 1.1, and the element d is taken to be $\dim Dj_m$, then the complexified algebra $\mathbb{C} \otimes \mathcal{L}_d$ (ν is specialized to \sqrt{q}) is isomorphic to \mathcal{C} . This is because both algebras are isomorphic to the same generalized q -Schur algebra of type \mathbf{D}_{m+2} . It will be very interesting to make a direct connection of the two algebras \mathcal{C} and \mathcal{L}_d in [Li10] and [Li12], respectively. Note that in type \mathbf{A}_n case, the algebra \mathcal{L}_d for certain d is shown in [Li12, Section 8] to be isomorphic to the q -Schur algebra.

3. A PARAMETRIZATION

3.1. If the ground field \mathbb{F}_q is replaced by its algebraic closure \mathbb{F} , the set X becomes an algebraic variety over \mathbb{F} . We shall fix a mistake in [Li10]. The dimension of $X_{\nu, \underline{c}}$ in [Li10, 4.1] is

$$\sum_{a=i,k} \nu_a(\nu_{j_1} - \nu_a) + \sum_{\beta=1}^m \nu_{j_\beta}(\nu_{j_{\beta+1}} - \nu_{j_\beta}) - \sum_{\beta=1}^m (c_\beta - c_{\beta+1})(\nu_{j_{\beta+1}} - \nu_{j_\beta} - c_{\beta+1} + c_\beta).$$

3.2. Let $\Psi_2 : Y \rightarrow X$ be the morphism defined in [Li10, 4.2]. We set

$$Y_{\nu, \underline{c}} = \Psi_2^{-1}(X_{\nu, \underline{c}}).$$

We have a stratification of Y :

$$Y = \sqcup_{\nu, \underline{c}} Y_{\nu, \underline{c}}.$$

Lemma 3.3. *The restriction $\Psi_{2, \underline{c}} : Y_{\nu, \underline{c}} \rightarrow X_{\nu, \underline{c}}$ of Ψ_2 to $Y_{\nu, \underline{c}}$ is a vector bundle of fiber dimension*

$$\sum_{\beta=1}^m (c_\beta - c_{\beta+1})(\nu_{j_{\beta+1}} - \nu_{j_\beta} - c_{\beta+1} + c_\beta).$$

Proof. We will use the following fact. Fix a decomposition $E = E_1 \oplus E_2$ of a vector space. Let F_1 and F_2 be a subspace of E_1 and E_2 , respectively. Let \mathcal{F} be the collection of all subspaces F in E such that $F \cap E_1 = F_1$ and $\pi_2(F) = F_2$ where $\pi_2 : E \rightarrow E_2$ is the natural projection. Then \mathcal{F} is isomorphic to the vector space $\text{Hom}(F_2, E_1/F_1)$. A bijection $\phi \mapsto F(\phi)$ of the two spaces is defined by

$$F(\phi) = \{f_1 + f_2 + \phi(f_2) \mid f_1 \in F_1, f_2 \in F_2\},$$

where we fix a decomposition $E = F_1 \oplus E_1/F_1 \oplus F_2 \oplus E_2/F_2$.

By using this fact, we see that the fiber $\Psi_2^{-1}(U)$ is the same as the collection of linear maps $(\phi_\beta)_{1 \leq \beta \leq m}$ in $\oplus_{\beta=1}^m \text{Hom}(V_{j_\beta}, D/U_{j_\beta})$ such that

- (a) $\bar{\sigma}(\mathcal{V}(\phi_\beta)) \subseteq U_{j_{\beta+1}};$
- (b) $V_{j_{\beta+1}} \subseteq \mathcal{V}(\phi_\beta) \cap 0 \oplus D, \quad \forall 1 \leq \beta \leq m.$

The condition (a) holds if and only if $\phi_\beta(v) \in U_{j_{\beta+1}}$ for $1 \leq \beta \leq m$. The condition (b) holds if and only if $\phi_\beta(v) = 0$ for any $v \in V_{j_{\beta+1}}$. Therefore, the fiber $\Psi_2^{-1}(U)$ is isomorphic to the vector space

$$\oplus_{\beta=1}^m \text{Hom}(V_{j_\beta}/V_{j_{\beta+1}}, U_{j_{\beta+1}}/U_{j_\beta}).$$

The Lemma follows. □

A consequence of Lemma 3.3 is that $Y_{\nu, \underline{c}}$ is smooth. From this Lemma, we have

$$\dim Y_{\nu, \underline{c}} = \dim X_{\nu, \underline{c}} + \sum_{\beta=1}^m (c_\beta - c_{\beta+1})(\nu_{j_{\beta+1}} - \nu_{j_\beta} - c_{\beta+1} + c_\beta) = \dim Y_\nu.$$

So we have

Proposition 3.4. *The irreducible components of Y_ν are the closure $\overline{Y_{\nu, \underline{c}}}$ of $Y_{\nu, \underline{c}}$ for any sequence $\underline{c} = (c_m, \dots, c_1)$ of non-decreasing and non-negative integers such that $c_1 \leq \min\{\nu_i, \nu_k\}$.*

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MATHEMATICS AND STATISTICS, LOYOLA UNIVERSITY CHICAGO, CHICAGO, IL 60626, USA
E-mail address: doty@math.luc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY AT BUFFALO, SUNY, 244 MATHEMATICS BUILDING,
 BUFFALO, NY 14260
E-mail address: yiqiang@buffalo.edu